Mathematical Structures for Computer Science

Discrete Mathematics and Its Applications

SEVENTH EDITION

JUDITH L. GERSTING

Mathematical Structures for Computer Science

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Edition 7

Mathematical Structures for Computer Science

DISCRETE MATHEMATICS AND ITS APPLICATIONS

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Indiana University-Purdue University at Indianapolis

W. H. Freeman and Company
A Macmillan Higher Education Company

To my 0110_2 favorite

discrete structures:

(Adam $\wedge$ Francine),

(Jason $\wedge$ Cathryn) $\rightarrow$

(Sammie $\wedge$ Johnny)

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Illustrations: Network Graphics
Composition: codeMantra
Printing and Binding: RR Donnelley

Library of Congress Control Number: 2013951442

ISBN-13: 978-1-4292-1510-7
ISBN-10: 1-4292-1510-0

Printed in the United States of America
First printing

W. H. Freeman and Company
41 Madison Avenue, New York, NY 10010
Houndmills, Basingstoke RG21 6XS, England
www.whfreeman.com

Contents in Brief

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Preface

A course in discrete structures (discrete mathematics) played an important role in Curriculum 68, the very first ACM Computer Science Curriculum Guide: "This course introduces the student to those fundamental algebraic, logical, and combinatoric concepts from mathematics needed in the subsequent computer science courses and shows the applications of these concepts to various areas of computer science."[1] Fast forward 45 years or so (through mobile computing, wireless networks, robotics, virtual reality, 3-D graphics, the Internet …) to the joint ACM/IEEE-CS Computer Science Curricula 2013, where—still—discrete structures are of fundamental importance. "The material in discrete structures is pervasive in the areas of data structures and algorithms but appears elsewhere in computer science as well. For example, an ability to create and understand a proof—either a formal symbolic proof or a less formal but still mathematically rigorous argument—is important in virtually every area of computer science, including (to name just a few) formal specification, verification, databases, and cryptography. Graph theory concepts are used in networks, operating systems, and compilers. Set theory concepts are used in software engineering and in databases. Probability theory is used in intelligent systems, networking, and a number of computing applications."[2]

This Seventh Edition was guided by Curricula 2013, and virtually all of the Core Tier 1 and Tier 2 topics for discrete structures from that document are included. Covering all those topics can fill a one-semester course, but there is certainly enough material in this edition to make for a very respectable two-semester course.

However much we as instructors may see the value in this foundational course, it is a difficult experience for many students, who often view it as a series of unconnected topics with little or no application to the rest of their chosen field of study. In the big picture, these topics are bound together by themes of

- importance of logical thinking
- power of mathematical notation
- usefulness of abstractions

but such themes are best appreciated in hindsight. Telling students, "You will need ideas from this course in many of your future computer science courses," is also of little motivation. That's why it is important to carve out time in your course syllabus (for either a one-semester or two-semester course) for some of the applications of this material. Here are topics in this edition that you may

[1]*Communications of the ACM,* Vol. 11, Issue 3 (March 1968), pp. 151–197.
[2]Computer Science Curricula 2013, Pre-release version, http://cs2013.com

choose from, according to your interests and the interests of your students. Yes, students will probably see most of these topics in more detail in later computer science courses, but a quick introduction now can keep their interest and make your claim of relevance more credible.

Section 1.5	Logic programming
Sections 1.6 and 2.3	Proof of correctness
Section 3.3	Analysis of algorithms
Section 5.3	Relations and databases
Section 5.6	The mighty mod function
Section 6.4	Huffman codes
Section 8.2	Logic networks
Section 9.2	Coding theory

In addition, there is a Special Interest Page in each chapter that highlights interesting applications culled from "the real world."

NEW IN THE SEVENTH EDITION

- The former Chapters 2 and 3 have been reorganized as Chapters 2, 3, and 4 for better clarity and sequencing
- New sections or subsections have been added:

  Probability
  - Bayes' Theorem
  - Binomial Distribution

  Order of Magnitude (new section)
  - The Master Theorem
  - Proof of the Master Theorem

  Matrices
  - Gaussian Elimination

  Coding Theory (new section)
  - Introduction
  - Background: Homomorphisms and Cosets
  - Generating Group Codes
  - Decoding Group Codes
- "Special interest pages"—one per chapter—have been introduced to add relevance and interest to the material being presented.
- Answers to all odd-numbered exercises, as opposed to answers to fewer, selected exercises, appear in the back of the book. When an exercise asks

for a proof, the complete proof is given. Otherwise, the answer is just the answer, not necessarily the solution. A Student Solutions Manual with solutions for odd-numbered exercises from the book is available from the Web site at www.whfreeman.com/gersting. The student manual also includes two sample tests per chapter. A complete Solutions Manual is available to instructors from the publisher.

- Many new exercises have been added, particularly with an eye toward pairing odd-numbered exercises with similar even-numbered exercises.
- Of course, student learning aids such as chapter objectives, practice problems, reminders, section reviews, and chapter reviews remain.

WEB SITE

Online Study Guide

A Web site for the book may be found at www.whfreeman.com/gersting. The Web pages contain representative new example problems (not contained in the text) for many of the end-of-section Techniques. Each Technique that has a corresponding Web page example is marked with the icon Ⓦ.

Each example on the Web first states the problem. Then succeeding pages develop the solution, much as the student would be expected to write it. As the student navigates the pages, the solution unfolds step-by-step. A compressed audio file is also part of each Web page after the initial problem statement. The audio file contains a first-person stream-of-consciousness thought process about that step of the solution—why it occurred to the narrator to try this, why it looked promising, what knowledge was being called on to suggest that this step should come next, and so on. The point is, students see perfect and complete worked-out proofs in the textbook and often see them performed by the instructor. Yet when a student goes home and tries to produce such a solution by himself or herself, he or she is unsure where to start or how to think about the problem or how to see any pattern to enable a guess as to what to do next. Consequently the student gives up in frustration. The purpose of the audio narration is to share the "secret picture" that mathematicians use to solve problems.

To access the problems, after you go to www.whfreeman.com/gersting, select a chapter section, then select a sample problem and follow its step-by-step process with the "Next" button.

PowerPoint Slides

Instructors who visit the web site will also have access to PowerPoint slides accompanying each section of the text.

ACKNOWLEDGMENTS

My thanks to the reviewers of this addition, as well to reviewers of earlier editions, all of whose help is greatly appreciated.

Elizabeth Adams, *James Madison University*
Kemal Akkaya, *Southern Illinois University*
Charles Ashbacher, *Mount Mercy College*
Barnabas Bede, *DigiPen Institute of Technology*
Terry J. Bridgeman, *Colorado School of Mines*
David Casperson, *University of Northern British Columbia*
Adrienne Decker, *SUNY Buffalo*
Steve Donaldson, *Samford University*
Mordechai S. Goodman, *Dominican University*
Michael A. Gray, *American University*
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Damian Lyons, *Fordham University*
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Mikel D. Petty, *University of Alabama in Huntsville*
Amar Raheja, *Cal Poly*
J. Ben Schafer, *University of Northern Iowa*
Ali Shaykhian, *Florida Institute of Technology*
Shunichi Toida, *Old Dominion University*
William J. Weber, *Southeast Missouri State University*
Eric Westlund, *Luther University*
Hua Yan, *Borough of Manhattan Community College*
Yu Zhang, *Texas A&M Corpus Christi*

The folks at W.H. Freeman were very helpful in shepherding this edition to completion, especially Penny Hull (veteran of many previous editions), Terri Ward, Roland Cheyney, Liam Ferguson, Georgia Lee Hadler, and Vicki Tomaselli.

Thanks to Russell Kackley for the audio files on the Web site.

My deepest thanks go to my husband, John, ever my most ardent supporter and dearest friend.

NOTE TO THE STUDENT

As you go through this book, you'll encounter many new terms and new ideas. Try reading with pencil and paper at hand and work the Practice problems as you encounter them. They are intended to reinforce or clarify some new terminology or method just introduced; answers are given at the back of the book. Pay attention also to the Reminders that point out common pitfalls or provide helpful hints.

Be sure to visit the Web site at www.whfreeman.com/gersting for detailed, worked-out solutions to additional example problems tied to the Techniques in each section. The Web site solutions are accompanied by audio files that explain each step. A Student Solutions Manual with solutions for odd-numbered exercises from the book is available from the Web site. The student manual also includes two sample tests per chapter.

You may find at first that the thought processes required to solve the exercises in the book are new and difficult. Your biggest attribute for success will be perseverance. Here's what I tell my students: "If you do not see at first how to solve a problem, don't give up, think about it some more; be sure you understand all the terminology used in the problem, play with some ideas. If no approach presents itself, let it be and think about it again later. Repeat this process for days on end. When you finally wake up in the middle of the night with an idea, you'll know you are putting in the right amount of effort for this course." Mathematical results don't spring fully formed from the foreheads of mathematical geniuses; well, maybe from mathematical geniuses, but for the rest of us, it takes work, patience, false starts, and **perseverance**.

Enjoy the experience!

Chapter 1

Formal Logic

CHAPTER OBJECTIVES

After studying this chapter, you will be able to:

- Use the formal symbols of propositional logic.
- Find the truth value of an expression in propositional logic.
- Construct formal proofs in propositional logic, and use such proofs to determine the validity of English language arguments.
- Use the formal symbols of predicate logic.
- Find the truth value in some interpretation of an expression in predicate logic.
- Use predicate logic to represent English language sentences.
- Construct formal proofs in predicate logic, and use such proofs to determine the validity of English language arguments.
- Understand how the programming language Prolog is built on predicate logic.
- Mathematically prove the correctness of programs that use assignment statements and conditional statements.

You have been selected to serve on jury duty for a criminal case. The attorney for the defense argues as follows:

> If my client is guilty, then the knife was in the drawer. Either the knife was not in the drawer or Jason Pritchard saw the knife. If the knife was not there on October 10, it follows that Jason Pritchard did not see the knife. Furthermore, if the knife was there on October 10, then the knife was in the drawer and also the hammer was in the barn. But we all know that the hammer was not in the barn. Therefore, ladies and gentlemen of the jury, my client is innocent.

Question: **Is the attorney's argument sound? How should you vote?**

It's much easier to answer this question if the argument is recast in the notation of formal logic. Formal logic strips away confusing verbiage and allows us to concentrate on the underlying reasoning being applied. In fact, formal logic—the subject of this chapter—provides the foundation for the organized, careful method of thinking that characterizes any reasoned activity—a criminal investigation, a scientific experiment, a sociological study. In addition, formal logic has direct applications in computer science. The last two sections of this chapter explore a programming language based on logic and the use of formal logic to verify the correctness of computer programs. Also, circuit logic (the logic governing computer circuitry) is a direct analog of the statement logic of this chapter. We will study circuit logic in Chapter 8.

SECTION 1.1 STATEMENTS, SYMBOLIC REPRESENTATION, AND TAUTOLOGIES

Formal logic can represent the statements we use in English to communicate facts or information. A **statement** (or **proposition**) is a sentence that is either true or false.

EXAMPLE 1 Consider the following:

a. Ten is less than seven.
b. Cheyenne is the capital of Wyoming.
c. She is very talented.
d. There are life forms on other planets in the universe.

Sentence (a) is a statement because it is false. Sentence (b) is a statement because it is true. Sentence (c) is neither true nor false because "she" is not specified; therefore (c) is not a statement. Sentence (d) is a statement because it is either true or false; we do not have to be able to decide which. ●

Connectives and Truth Values

In English, simple statements are combined with connecting words like *and* to make more interesting compound statements. The truth value of a compound statement depends on the truth values of its components and which connecting words are used. If we combine the two true statement, "Elephants are big," and, "Baseballs are round," we would consider the resulting statement, "Elephants are big and baseballs are round," to be true. In this book, as in many logic books, capital letters near the beginning of the alphabet, such as A, B, and C, are used to represent statements and are called **statement letters**; the symbol $\wedge$ is a **logical connective** representing *and*. We agree, then, that if A is true and B is true, $A \wedge B$ (read "A and B") should be considered true.

PRACTICE 1 [1]

a. If A is true and B is false, what truth value would you assign to $A \wedge B$?
b. If A is false and B is true, what truth value would you assign to $A \wedge B$?
c. If A and B are both false, what truth value would you assign to $A \wedge B$? ■

The expression $A \wedge B$ is called the **conjunction** of A and B, and A and B are called the **conjuncts** of this expression. Table 1.1 summarizes the truth value of $A \wedge B$ for all possible truth values of the conjuncts A and B. Each row of the table

[1]Answers to practice problems are in the back of the text.

TABLE 1.1

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 1.2

A	B	$A \vee B$
T	T	T
T	F	
F	T	
F	F	

represents a particular truth value assignment to the statement letters, and the resulting truth value for the compound expression is shown.

Another connective is the word *or*, denoted by the symbol $\vee$. The expression $A \vee B$ (read "A or B") is called the **disjunction** of A and B, and A and B are called the **disjuncts** of this expression. If A and B are both true, then $A \vee B$ would be considered true, giving the first line of the truth table for disjunction (Table 1.2).

PRACTICE 2 Use your understanding of the word *or* to complete the truth table for disjunction, Table 1.2. ■

Statements may be combined in the form "if statement 1, then statement 2." If A denotes statement 1 and B denotes statement 2, the compound statement would be denoted by $A \rightarrow B$ (read "A implies B"). The logical connective here is **implication**, and it conveys the meaning that the truth of A implies or leads to the truth of B. In the implication $A \rightarrow B$, A stands for the **antecedent** statement and B stands for the **consequent** statement.

The truth table for implication is less obvious than that for conjunction or disjunction. To understand its definition, let's suppose your friend remarks, "If I pass my economics test, then I'll go to the movie Friday." If your friend passes the test and goes to the movie, the remark was true. If your friend passes the test but doesn't go to the movie, the remark was false. If your friend doesn't pass the test, then—whether he or she goes to the movie or not—you could not claim that the remark was false. You would probably want to give the benefit of the doubt and say that the statement was true. By convention, $A \rightarrow B$ is considered true if A is false, regardless of the truth value of B.

PRACTICE 3 Summarize this discussion by writing the truth table for $A \rightarrow B$. ■

The **equivalence** connective is symbolized by $\leftrightarrow$. Unlike conjunction, disjunction, and implication, the equivalence connective is not really a fundamental connective but a convenient shortcut. The expression $A \leftrightarrow B$ stands for $(A \rightarrow B) \wedge (B \rightarrow A)$. We can write the truth table for equivalence by constructing, one piece at a time, a table for $(A \rightarrow B) \wedge (B \rightarrow A)$, as in Table 1.3. From this truth table, $A \leftrightarrow B$ is true exactly when A and B have the same truth value.

TABLE 1.3

A	B	$A \rightarrow B$	$B \rightarrow A$	$(A \rightarrow B) \wedge (B \rightarrow A)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The connectives we've seen so far are called **binary connectives** because they join two expressions together to produce a third expression. Now let's consider a **unary connective**, a connective acting on one expression to produce a second expression. **Negation** is a unary connective. The negation of A—symbolized by A'—is read "not A."

PRACTICE 4 Write the truth table for A'. (It will require only two rows.) ■

Table 1.4 summarizes the truth values for all of the logical connectives. This information is critical to an understanding of logical reasoning.

TABLE 1.4

A	B	$A \wedge B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$	A'
T	T	T	T	T	T	F
T	F	F	T	F	F	
F	T	F	T	T	F	T
F	F	F	F	T	T	

Because of the richness of the English language, words that have different shades of meaning are nonetheless represented by the same logical connective. Table 1.5 shows the common English words associated with various logical connectives.

REMINDER
A only if *B* means $A \rightarrow B$

TABLE 1.5

English Word	Logical Connective	Logical Expression
and; but; also; in addition; moreover	Conjunction	$A \wedge B$
or	Disjunction	$A \vee B$
If *A*, then *B*. *A* implies *B*. *A*, therefore *B*. *A* only if *B*. *B* follows from *A*. *A* is a sufficient condition for *B*. *B* is a necessary condition for *A*.	Implication	$A \rightarrow B$
A if and only if *B*. *A* is necessary and sufficient for *B*.	Equivalence	$A \leftrightarrow B$
not *A* It is false that *A* ... It is not true that *A* ...	Negation	A'

Suppose that $A \rightarrow B$ is true. Then, according to the truth table for implication, the consequent, B, can be true even though the antecedent, A, is false. So while the truth of A leads to (implies) the truth of B, the truth of B does not imply the truth of A. The phrase "B is a necessary condition for A" to describe $A \rightarrow B$ simply means that if A is true, then B is necessarily true, as well. "A only if B" describes the same thing, that A implies B.

EXAMPLE 2 The statement, "Fire is a necessary condition for smoke," can be restated, "If there is smoke, then there is fire." The antecedent is "there is smoke," and the consequent is "there is fire." ●

PRACTICE 5 Name the antecedent and consequent in each of the following statements. (*Hint*: Rewrite each statement in if-then form.)

a. If the rain continues, then the river will flood.
b. A sufficient condition for network failure is that the central switch goes down.
c. The avocados are ripe only if they are dark and soft.
d. A good diet is a necessary condition for a healthy cat. ■

EXAMPLE 3 Expressing the negation of a statement must be done with care, especially for a compound statement. Table 1.6 gives some examples. ●

TABLE 1.6

Statement	Correct Negation	Incorrect Negation
It will rain tomorrow.	It is false that it will rain tomorrow. It will not rain tomorrow.	
Peter is tall and thin.	It is false that Peter is tall and thin. Peter is not tall or he is not thin. Peter is short or fat.	Peter is short and fat. Too strong a statement. Peter fails to have both properties (tallness and thinness) but may still have one property.
The river is shallow or polluted.	It is false that the river is shallow or polluted. The river is neither shallow nor polluted. The river is deep and unpolluted.	The river is not shallow or not polluted. Too weak a statement. The river fails to have either property, not just fails to have one property.

PRACTICE 6 Which of the following represents A' if A is the statement "Julie likes butter but hates cream"?

a. Julie hates butter and cream.
b. Julie does not like butter or cream.
c. Julie dislikes butter but loves cream.
d. Julie hates butter or likes cream. ■

We can string statement letters, connectives, and parentheses (or brackets) together to form new expressions, as in

$$(A \rightarrow B) \wedge (B \rightarrow A)$$

Of course, just as in a computer programming language, certain *syntax rules* (rules on which strings are legitimate) prevail; for example,

$$A\ )) \wedge\wedge \rightarrow BC$$

would not be considered a legitimate string. An expression that is a legitimate string is called a **well-formed formula**, or **wff**. To reduce the number of parentheses required in a wff, we stipulate an order in which connectives are applied. This *order of precedence* is

1. connectives within parentheses, innermost parentheses first
2. $'$
3. $\wedge, \vee$
4. $\rightarrow$
5. $\leftrightarrow$

This means that the expression $A \vee B'$ stands for $A \vee (B')$, not $(A \vee B)'$. Similarly, $A \vee B \rightarrow C$ means $(A \vee B) \rightarrow C$, not $A \vee (B \rightarrow C)$. However, we often use parentheses anyway, just to be sure that there is no confusion.

In a wff with a number of connectives, the connective to be applied last is the **main connective**. In

$$A \wedge (B \rightarrow C)'$$

the main connective is $\wedge$. In

$$((A \vee B) \wedge C) \rightarrow (B \vee C')$$

the main connective is $\rightarrow$. Capital letters near the end of the alphabet, such as P, Q, R, and S, are used to represent wffs. Thus P could represent a single statement letter, which is the simplest kind of wff, or a more complex wff. We might represent

$$((A \vee B) \wedge C) \rightarrow (B \vee C')$$

as

$$P \rightarrow Q$$

if we want to hide some of the details for the moment and only concentrate on the main connective.

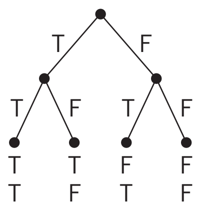
Wffs composed of statement letters and connectives have truth values that depend on the truth values assigned to their statement letters. We write the truth table for any wff by building up the component parts, just as we did for $(A \rightarrow B) \wedge (B \rightarrow A)$. The main connective is addressed in the last column of the table.

EXAMPLE 4 The truth table for the wff $A \vee B' \rightarrow (A \vee B)'$ is given in Table 1.7. The main connective, according to the rules of precedence, is implication. ●

TABLE 1.7

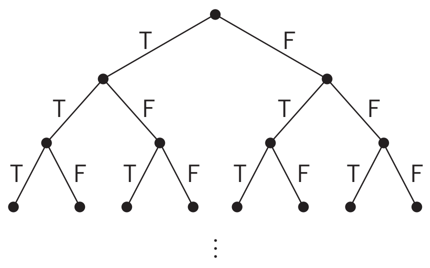
A	B	B'	$A \vee B'$	$A \vee B$	$(A \vee B)'$	$A \vee B' \rightarrow (A \vee B)'$
T	T	F	T	T	F	F
T	F	T	T	T	F	F
F	T	F	F	T	F	T
F	F	T	T	F	T	T

If we are making a truth table for a wff that contains n different statement letters, how many rows will the truth table have? From truth tables done so far, we know that a wff with only one statement letter has two rows in its truth table, and a wff with two statement letters has four rows. The number of rows equals the number of true-false combinations possible among the statement letters. The first statement letter has two possibilities, T and F. For each of these possibilities, the second statement letter has two possible values. Figure 1.1a pictures this as a two-level "tree" with four branches showing the four possible combinations of T and F for two statement letters. For n statement letters, we extend the tree to n levels, as in Figure 1.1b. The total number of branches then equals 2^n. The total number of rows in a truth table for n statement letters is also 2^n.



Statement letters	Choices
1	$2 = 2^1$ branches
2	$4 = 2^2$ branches

(a)



Statement letters	Choices
1	$2 = 2^1$ branches
2	$4 = 2^2$ branches
3	$8 = 2^3$ branches
⋮	⋮
n	2^n branches

Figure 1.1 (b)

TABLE 1.8

A	*B*	*C*
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

This tree structure also tells us how to enumerate all the T–F combinations among the n statement letters when setting up a truth table. If we read each level of the tree from bottom to top, it says that the T–F values for statement letter n (which will compose the last column of the truth table) alternate, those for statement letter $n - 1$ alternate every two values, those for statement letter $n - 2$ alternate every four values, and so forth. Thus a truth table for three statement letters would begin as shown in Table 1.8. The values for statement letter C alternate, those for statement letter B alternate in groups of two, and those for statement letter A alternate in groups of four, resulting in something like a sideways version of the tree. (Reading the rows from the bottom up and using 1 for T and 0 for F shows that we are simply counting up from zero in binary numbers.)

PRACTICE 7 Construct truth tables for the following wffs.

a. $(A \rightarrow B) \leftrightarrow (B \rightarrow A)$ (Remember that $C \leftrightarrow D$ is true precisely when C and D have the same truth value.)
b. $(A \vee A') \rightarrow (B \wedge B')$
c. $[(A \wedge B') \rightarrow C']'$
d. $(A \rightarrow B) \leftrightarrow (B' \rightarrow A')$ ■

Tautologies

A wff-like item (d) of Practice 7, whose truth values are always true, is called a **tautology**. A tautology is "intrinsically true" by its very structure; it is true no matter what truth values are assigned to its statement letters. A simpler example of a tautology is $A \vee A'$; consider, for example, the statement "Today the sun will shine or today the sun will not shine," which must always be true because one or the other of these must happen. A wff like item (b) of Practice 7, whose truth values are always false, is called a **contradiction**. A contradiction is "intrinsically false" by its very structure. A simpler example of a contradiction is $A \wedge A'$; consider "Today is Tuesday and today is not Tuesday," which is false no matter what day of the week it is.

REMINDER

A, *B*, *C* stand for single statement letters; *P*, *Q*, *R*, *S* stand for wffs.

Suppose that P and Q represent two wffs, and it happens that the wff $P \leftrightarrow Q$ is a tautology. If we did a truth table using the statement letters in P and Q, then the truth values of the wffs P and Q would agree for every row of the truth table. In this case, P and Q are said to be **equivalent wffs**, denoted by $P \Leftrightarrow Q$. Thus $P \Leftrightarrow Q$ states a fact, namely, that the particular wff $P \leftrightarrow Q$ is a tautology. Practice 7(d) has the form $P \leftrightarrow Q$, where P is the wff $(A \rightarrow B)$ and Q is the wff $(B' \rightarrow A')$, and $P \leftrightarrow Q$ was shown to be a tautology. Therefore, $(A \rightarrow B) \Leftrightarrow (B' \rightarrow A')$.

We will list some basic equivalences, prove one or two of them by constructing truth tables, and leave the rest as exercises. We represent any contradiction by 0 and any tautology by 1.

Some Tautological Equivalences

1a. $A \vee B \Leftrightarrow B \vee A$	1b. $A \wedge B \Leftrightarrow B \wedge A$	(commutative properties)
2a. $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$	2b. $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$	(associative properties)
3a. $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$	3b. $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$	(distributive properties)
4a. $A \vee 0 \Leftrightarrow A$	4b. $A \wedge 1 \Leftrightarrow A$	(identity properties)
5a. $A \vee A' \Leftrightarrow 1$	5b. $A \wedge A' \Leftrightarrow 0$	(complement properties)

Note that 2a allows us to write $A \vee B \vee C$ with no need for parentheses because the grouping doesn't matter; similarly, 2b allows us to write $A \wedge B \wedge C$.

EXAMPLE 5 The truth table in Table 1.9a verifies equivalence 1a, the commutative property for disjunction, and that in Table 1.9b verifies 4b, the identity property for conjunction. Note that only two rows are needed for Table 1.9b because 1 (a tautology) cannot take on false truth values. ●

TABLE 1.9

(a)

A	B	$A \vee B$	$B \vee A$	$A \vee B \leftrightarrow B \vee A$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

(b)

A	1	$A \wedge 1$	$A \wedge 1 \leftrightarrow A$
T	T	T	T
F	T	F	T

PRACTICE 8 Verify equivalence 5a. ■

The equivalences in the list are grouped into five pairs. In each pair, one equivalence can be obtained from the other by replacing $\wedge$ with $\vee$, $\vee$ with $\wedge$, 0 with 1, or 1 with 0. Each equivalence in a pair is called the **dual** of the other. Thus, 1a and 1b (commutativity of disjunction and commutativity of conjunction) are duals of each other. This list of equivalences appears in a more general setting in Chapter 8.

Two additional equivalences that are very useful are **De Morgan's laws**, named for the nineteenth-century British mathematician Augustus De Morgan, who first stated them. This theorem is easy to prove (see Exercises 26e and 26f).